Tutorial Assignment 3: Phonons, Dilution and Magnetic Cooling

Write clearly. Answer to 4 significant figures. Marks given for units and explanation of calculations.

1. In a solid, atoms move in coordinated motion. Their displacements can be described by sinusoidal waves, just like wave functions of particles in an ideal gas.

   i) Using the density of states of an ideal gas, derive the density of states $g(k)$ of the waves in the solid. k the wavevector. [2]

   ii) State the relation between k and angular frequency ω of a wave. Transform $g(k)$ to $g(\omega)$. [2]

   iii) Why is there an upper limit to the frequency? State the name of this frequency. [2]

   iv) Assume that a solid with N atoms has $3N$ frequency states. Write down an equation in terms of $g(\omega)$ for the total number of states. [2]

   v) Using (iv), solve for the upper limit to the frequency. [2]

   vi) Assume that the waves are quantised into particles. State the name of this particle and its energy at ω. [2]

   vii) State the formula for the frequency distribution $f(\omega)$ of these particles. Why is the chemical potential μ absent in this formula? [2]

   viii) From (ii) and (vii), derive an integral expression for the total energy U of these particles. [2]

   ix) In the high T limit, obtain an approximate integral expression for the integral in (viii). [2]

   x) Integrate (ix) to find the total energy in the high temperature limit. Find the heat capacity for 1 mol of the solid. [2]

2. i) One mole of ^{3}He is mixed with 1 mol of ^{4}He in a container at 2 K. Describe what happens when this mixture is cooled down to 10 mK. [2]

   ii) Find the Fermi temperature of ^{3}He in the upper layer. Assume that the effective mass ratio is 2.8 and that 1 mol of liquid helium is 30 cm^3. [2]

   iii) Find the Fermi temperature of ^{3}He in the lower layer. Assume that the effective mass ratio is 2.4. [2]

iv) Find the heat capacities of 1 mol of ^{3}He in (i) and in (ii). How much heat is absorbed if 0.1 mol of ^{3}He diffuses from upper to lower layer? [2]

v) From tutorial 2, the number of excited electrons is $3g(E_F)k_BT$. For the same number of ^{3}He atoms, in which layer is $g(E_F)$ higher? Hence explain why heat is absorbed. [2]

3. A spin ½-salt sits in a magnetic field B and temperature T. It contains 0.2 mol of magnetic ions. Each ion has 2 magnetic energy levels.

i) When this salt is cooled down from a high temperature to 1 K, half of the ions in the higher level fall to the lower level. Find the new population in each level in moles. [2]

ii) Find the energy difference between the two levels and the heat given out in (i). What is the strength of the magnetic field? [2]

iii) Suppose that we now thermally insulate the salt from the surrounding. Why would B/T be constant? If we reduce B by 90%, what is the final T? [2]

iv) For this final T, find the cooling power of the salt. What limits the lowest temperature it can reach and why? [2]

v) Sketch an entropy versus temperature graph of the above cycle. Label the 3 main paths of the cycle. Shade and label the 2 areas that are equal to the heats in (ii) and (iv). [2]